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LETTER TO THE EDITOR

Critical ultrasonics near the superfluid transition: finite-size effects

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Abstract. The suppression of order parameter fluctuations at the boundaries causes the ultrasonic attenuation near the superfluid transition to be lowered below the bulk value. We calculate explicitly the first deviation from the bulk value for temperatures above the lambda point. This deviation is significantly larger than for static quantities such as the thermodynamic specific heat or other transport properties such as the thermal conductivity. This makes ultrasonics a very effective probe for finite-size effects.

Critical phenomena in confined geometry have been attracting a fair amount of attention of late [1] because of the progress on the experimental front [2–9] which is making it possible to check the predictions of finite-size effects (FSE). A fair amount of this experimental effort has gone into studying the specific heat near the superfluid transition. With the bulk-specific heat quite well understood and the existence of a sharp phase transition (apart from gravity rounding, which can also be removed by carrying out experiments in space) established, efforts have been made to study the FSE. It is expected that the FSE will round out the transition and hence the divergence at $T = T_\lambda$ will be removed. The specific heat will be finite and the finite value will be a function of the confining length. We will keep in mind one of the favoured experimental geometries, where one takes two parallel plates separated by a distance L, much smaller than the linear dimensions of the plates. For $L \gg \xi$, the correlation length at a given temperature, the usual thermodynamic result follows. It is when $L \leqslant \xi$, that FSE dominate. Finite-size scaling suggests the existence of a scaling function, function of ξ/L in terms of which the theory can be cast. The specific heat $C(t, L)$ in finite geometry has the form $C(t, L) \sim t^{-\alpha} g(t^{-\nu}/L) +$ constant where $\xi \sim t^{-\nu}$ and $t = (T - T_\lambda)/T_\lambda$,$T_\lambda$ being the transition temperature. The function g has been calculated by various authors [10–12]. In what follows we propose a method of checking for FSE by studying a related dynamic property. This is the study of ultrasonic attenuation (UA) near t_λ at high frequencies. In fact, it is our contention that UA is one of the best ways of checking for FSE since the single surface effect alone can produce effects greater than 10%. The critical fluctuations relax according to ξ^{-z}, where z is the dynamic scaling exponent. For frequencies ω such that $\omega \gg \Gamma_0$, the Onsager's constant (occurs if one is close to the critical point), the attenuation is independent of the correlation length. For a finite-size system, we shall show that in this limit, the attenuation is determined by ω and L alone. We provide explicit answers for frequencies ω which are much smaller than a cut-off frequency ω_0 (of the order of a few GHz) and for plate separation $L \geqslant (2\Gamma_0/\omega)^{1/2}$

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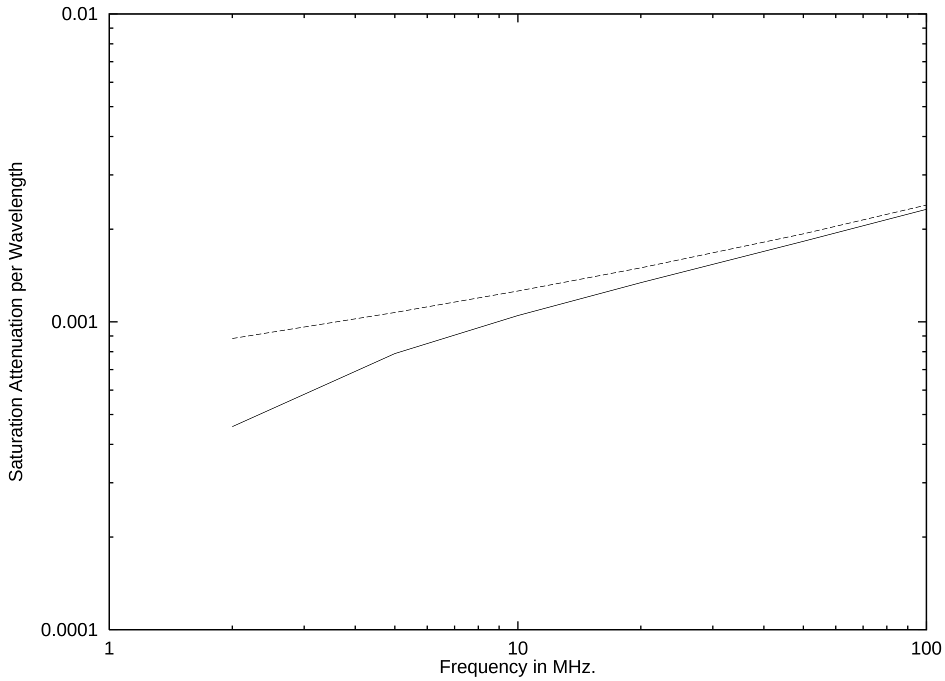

Figure 1. Saturation attenuation is plotted against frequency. The broken curve shows the bulk ($L \to \infty$) result whereas the full curve shows the surface effect.

for a given frequency ω. Our prediction for the attenuation per wavelength as a function of ω for the plate separation of 2110 Å is shown in figure 1. It should be possible to check the prediction experimentally. In fact, this should be the simplest way of checking for FSE since the effect is quite pronounced (about 18% at $L = 2110$ Å and $\omega = 10$ Mhz) for the available confining lengths as shown in figure 1. This occurs because the imaginary part of the specific heat determines the UA and is much smaller than the real part, but as we shall see below, *both are equally affected by the FSE*. Consequently, the relative effect is much larger for the imaginary part and this will show up in the UA.

The basis of our calculation is once more the Pippard–Buckingham–Fairbank (PBF) relation [13, 14] which gives a successful account [15–17] of the critical ultrasonics in the situation where $L \gg \xi$. The PBF relation is obtained from general considerations of entropy clamping and yields for the sound velocity $u(T, \omega)$

$$u(T, \omega) = u_0(T_0) + u_1 C_0 / C_P(T, \omega) \tag{1}$$

where $u_0(T_0)$ is the sound speed at the transition point (T_0 is the bulk T_λ for the infinite system, but is a L-dependent temperature for the finite-size system), u_1 and C_0 are constants and $C_P(T, \omega)$ is the specific heat at finite frequency.

For the bulk case, $C_P(T, \omega = 0)$ diverges at $T = T_\lambda$ and $C_P(T, \omega)$ is a homogeneous function of ω and ξ. If the characteristic relaxation rate is $\Gamma_0 \xi^{-z}$ then the scaling form of C_P^{bulk} is

$$C_P^{\mathrm{bulk}}(T, \omega) = \xi^{\alpha/\nu} f\left(\frac{\omega}{\Gamma_0 \xi^{-z}}\right). \tag{2}$$

The exponent α is very close to zero for the superfluid transition in ^{4}He and for many practical purposes, it is possible to write

$$C_P^{\mathrm{bulk}}(T,\omega) = C\left[\ln(\Lambda\xi) + f\left(\frac{\omega}{\Gamma_0\xi^{-z}}\right)\right]. \tag{3}$$

The function $f(\omega/(\Gamma_0\xi^{-z}))$ reduces to a constant for $\omega = 0$ and tends to $-\ln(\omega/\Gamma_0)^{1/z}\xi$ for $\omega \gg \Gamma_0\xi^{-z}$. A one-loop calculation of the scaling function $f(\Omega)$ where $\Omega = \omega/\Gamma_0\xi^{-z}$, was carried out and led to a successful scaling theory of the attenuation in the bulk ^{4}He near T_λ [15–17].

We now need to discuss the effect of a confining geometry. At zero frequency, the specific heat is blunted due to the FSE and the usually divergent specific heat remains finite. The single-loop calculation of the scaling function $g(\xi/L)$ discussed before gives a very reasonable account of the recent specific heat data by Mehta and Gasparini [2]. One of the most important feature of the scaling function is the low ξ/L limit (experimentally most easily accessible) is the first departure from the thermodynamic limit—the magnitude of this departure ΔC has to be proportional to the surface-to-volume ratio $(A{:}V)$ and hence from purely dimensional arguments, the correction can be written as

$$\Delta C = C(\xi, L) - C_\infty(\xi) = -aCA\frac{\xi}{V} \tag{4}$$

where a is a number of O(1), which can be obtained from the function $g(\xi/L)$, and C is the dimensional constant defined in equation (3). The value of a as inferred from Schmolke *et al* [11] is 1.4. The agreement of this departure with the measured departure of Mehta and Gasparini is impressive.

For our present concern we need the three variable functions $C(\xi, L, \omega)$, whose two limits $C(\xi,\omega)$ and $C(\xi, L)$ are already well known. We will characterize $C(\xi, L, \omega)$ by its first departure from the infinite-volume limit $C(\xi,\omega)$ and write the generalization of equation (4) as

$$\Delta C(\xi, L, \omega) = C(\xi, L, \omega) - C(\xi,\omega) = -a(\xi,\omega)C(\xi)A/V \tag{5}$$

where $a(\xi,\omega)$ is a scaling function, whose zero-frequency limit is $a\xi$ (see equation (4)) and whose general form will be presented below. As soon as we start discussing the scaling function for $C(\xi, L, \omega)$ we need to worry about what sets the scale for ω. As we have discussed above, this has to be the rate of decay of fluctuations $\Gamma(\xi)$. In the finite geometry that we are now discussing, the scale for decay of fluctuations will also depend on L. In discussing the correction depicted in equation (5), it is obvious that this fine point does not need to be discussed as this correction is already $\mathrm{O}(\frac{1}{L})$. For He (superfluid transition), there is in someways an additional simplifying feature. For the order parameter decay rate the nonlinear effect of fluctuations becomes significant, only very close to the critical point and for all practical purposes, the relaxation rate can be taken to be at its noncritical background value. This means the dynamic critical exponent z can be taken to be 2.

The complex order parameter field $\psi_i(x)$ $\{i = 1, 2\}$ will be governed by the Langevin equation

$$\dot{\psi}_i = -\Gamma_0\frac{\delta F}{\delta\psi_i} + N_i \tag{6}$$

where

$$F = \int \mathrm{d}^D x \left[\frac{m^2}{2}\psi^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{\lambda}{4}(\psi^2)^2\right] \tag{7}$$

and N is a Gaussian white noise. The Gaussian white noise ensures that the fluctuation–dissipation theorem will hold and the equilibrium correlation function of the theory will be obtainable from the free energy functional of equation (7). For reasons stated above we choose to drop the reversible term (the Josephson equation for the phase of the order parameter). The parameter m^2 is proportional to $T - T_\lambda$, where T_λ is the bulk transition temperature. The system is confined in one of the D directions. We call that the z-direction. It is convenient to work with the Fourier transform in $D-1$ directions and the Fourier series (Dirichlet boundary conditions at $z = 0$ and $z = L$ suppressing the fluctuations) in the z-direction. The expansion of the time-dependent order parameter field is

$$\psi_i(\boldsymbol{r}, t) = \sum_n \psi_i(n, K, t) \exp^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R}} \sin\left(\frac{n\pi z}{L}\right). \tag{8}$$

The equation of motion for $\psi_i(n, K, t)$ is

$$\dot{\psi}_i(n, K, t) = -\Gamma_0 \left(m^2 + K^2 + \frac{n^2\pi^2}{L^2}\right) \psi_i(n, K, t) + N_i + \mathrm{O}(\psi^3). \tag{9}$$

In what follows, we will assume that all static correlations have been accounted for and $m^2 = \xi^{-2}$. The specific heat is obtained as the response function corresponding to the time-dependent correlation function

$$D(\xi, L, t_{12}) = \frac{1}{V} \iiint \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}^D R_{12}\, \langle \psi^2(\boldsymbol{R}_1, z_1, t_1) \psi^2(\boldsymbol{R}_2, z_2, t_2) \rangle \tag{10}$$

with $D(\xi, L, \omega) = 2\frac{\mathrm{Im}\, C(\xi, L, \omega)}{\omega}$ according to the fluctuation–dissipation theorem, straightforward algebra leads to the one-loop response function

$$\begin{aligned} C(\xi, L, \omega) &= \frac{1}{L} \sum_{\pm 1, \pm 2, \ldots} \int \frac{\mathrm{d}^{D-1} p}{(2\pi)^{D-1}} \frac{1}{(p^2 + m^2 + \frac{n^2\pi^2}{L^2})} \frac{1}{(-\frac{\mathrm{i}\omega}{2\Gamma_0} + p^2 + m^2 + \frac{n^2\pi^2}{L^2})} \\ &= \frac{1}{L} \sum_{0, \pm 1, \pm 2, \ldots} \int \frac{\mathrm{d}^{D-1} p}{(2\pi)^{D-1}} \frac{1}{(p^2 + m^2 + \frac{n^2\pi^2}{L^2})} \frac{1}{(-\frac{\mathrm{i}\omega}{2\Gamma_0} + p^2 + m^2 + \frac{n^2\pi^2}{L^2})} \\ &\quad - \frac{1}{L} \int \frac{\mathrm{d}^{D-1} p}{(2\pi)^{D-1}} \frac{1}{(p^2 + m^2)} \frac{1}{(-\frac{\mathrm{i}\omega}{2\Gamma_0} + p^2 + m^2)}. \end{aligned} \tag{11}$$

The sum in the first term on the right-hand side of equation (11) can be evaluated by the standard techniques of residues. If we are working to two-term accuracy, i.e. the bulk limit ($L \to \infty$) and the surface term (i.e. L^{-1}) then the result of working out the sum is the same as working out an integral ($L \to \infty$ makes the sum continuous) and thus

$$\begin{aligned} C(\xi, L, \omega) &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)} \frac{1}{(-\frac{\mathrm{i}\omega}{2\Gamma_0} + p^2 + m^2)} \\ &\quad - \frac{1}{L} \int \frac{\mathrm{d}^{D-1} p}{(2\pi)^{D-1}} \frac{1}{(p^2 + m^2)} \frac{1}{(-\frac{\mathrm{i}\omega}{2\Gamma_0} + p^2 + m^2)}. \end{aligned} \tag{12}$$

We work to logarithmic accuracy and hence evaluate the integrals at $D = 4$ (proper exponentiation can be undertaken by working to two-loop order, the details of which will be published elsewhere) to obtain the functions $f(\Omega)$ and $a(\Omega)$ introduced in equations (3) and (5). Note that since we are taking the logarithmic divergence for the bulk specific heat, the $C(\xi)$ in equations (4) and (5) reduces the constant C of equation (3). The function $f(\Omega)$ and $a(\Omega)$ are

$$f(\Omega) = \frac{1}{2} \left(\frac{1}{-\mathrm{i}\Omega} - 1\right) \ln(1 - \mathrm{i}\Omega) \tag{13}$$

$$a(\Omega) = \frac{\pi}{2} \frac{1}{-\mathrm{i}\Omega} \left[\sqrt{1 - \mathrm{i}\Omega} - 1\right] \tag{14}$$

leading to

$$
\begin{aligned}
C(\xi, L, \omega) = C_0 \Bigg\{ & \ln\frac{\Lambda}{m} - \frac{1}{4}\ln(1+\Omega^2) - \frac{1}{2\Omega}\tan^{-1}(\Omega) \\
& + \mathrm{i}\left[\frac{1}{2}\tan^{-1}(\Omega) - \frac{1}{4\Omega}\ln(1+\Omega^2)\right] - \frac{\pi}{mL\Omega}(1+\Omega^2)^{\frac{1}{4}}\sin\left(\frac{\tan^{-1}\Omega}{2}\right) \\
& - \frac{\mathrm{i}\pi}{mL\Omega}\left[(1+\Omega^2)^{\frac{1}{4}}\cos\left(\frac{\tan^{-1}\Omega}{2}\right) - 1\right]\Bigg\} \\
= {} & C_R + \mathrm{i}C_I
\end{aligned}
\tag{15}
$$

where C_R and C_I are the real and imaginary parts of the specific heat.

We now return to equation (1), to find the attenuation and dispersion. The *attenuation per wavelength* is

$$
\frac{\alpha\lambda}{2\pi} = \frac{u_1 C_0 C_I}{U_0(C_R^2 + C_I^2)}
$$

which leads to the frequency attenuation ($\omega \gg 2\Gamma_0 m^2$) as

$$
\frac{\alpha\lambda}{2\pi} = \frac{\pi u_1}{u_0}\frac{[1 - 2\sqrt{2}(\frac{2\Gamma_0}{\omega L^2})^{\frac{1}{2}}]}{[\ln(\frac{\omega_0}{\omega}) - \sqrt{2}\pi(\frac{2\Gamma_0}{\omega L^2})^{\frac{1}{2}}]^2 + \frac{\pi^2}{4}[1 - 2\sqrt{2}(\frac{2\Gamma_0}{\omega L^2})^{\frac{1}{2}}]^2}. \tag{16}
$$

This is the *saturation attenuation* per wavelength, which does not change as the temperature is lowered further, where $\omega_0/2\pi = 30$ GHz, $\Gamma_0 = 1.2 \times 10^{-4}$ cm^2 s^{-1}, $u_1/u_0 = \frac{8}{3} \times 10^{-2}$.

For the plate separation of 2110 Å of Mehta and Gasparini, the reduction in the attenuation due to the quenching of fluctuations is about 18% at 10 MHz and increases to 45% at 2.5 MHz. This is a large effect compared with the 4% surface effects that show up in the static measurements. For the corresponding measurement of thermal conductivity near the superfluid transition. Kahn and Ahlers [9] found that the deviation from the bulk is about 7% when the correlation length ξ equals the confining length L (in their case the radius of the pore). The surface effect for the ultrasonic measurement can easily amount to 30% which makes this an attractive system for a confrontation between theory and experiment. The effect of the finite size on the dispersion can be obtained from the real part of equation (1).

We note that the above is a one-loop calculation in the critical region. The lack of crossover to the background in our treatment of the specific heat implies that we can consider frequencies ω which are much smaller than the cut-off frequency ω_0. This is a restriction on the validity of the broken curve shown in figure 1. The full curve in addition is restricted to confining lengths which are not too small, i.e. $L \geqslant (2\Gamma_0/\omega)^{1/2}$ and in this regime the accuracy of the calculation is restricted by the loop order. This is not too severe a restriction as an accuracy of $\mathrm{O}(\epsilon)$ which our calculation entails, becomes an accuracy of $\mathrm{O}(\frac{\alpha}{\nu})$ when the combinatorial factors are included. Thus, in the above-mentioned ranges of the parameters ω and L, the broken curve in figure 1 should be an accurate prediction. It should be noted that in contrast to the static specific heat or the thermal conductivity which can be probed in real experiments as well as in computer simulations, the sound properties can only be probed in a real experiment. The final issue is then whether the effect can be observed in a real experiment. The critical ultrasonics near the superfluid transition was studied more than two decades ago. The most accurate data of that period lie in the 0.5–5 MHz range. In this region the scatter in the data is about 15%. This is somewhat better than the borderline for detecting the suppression reported here. Considering the fact that developments in the experimental field would enable more accurate measurements at present, we believe that this effect should be experimentally accessible.

The other sensitive part of an ultrasonic measurement is the low-frequency end ($\omega \ll 2\Gamma_0 m^2$), where for the bulk substance the attenuation per wavelength is proportional to $C_R^2\Omega/4$. The relative correction for the FSE is $1-\frac{\pi}{2mL}$, once again a larger effect than can be obtained in statics. For an easily realizable situation of $ml \sim 8$ this gives a 20% reduction in the attenuation. The whole course of the attenuation function with its dependence on ω and L is straightforward to obtain and will be exhibited elsewhere. Here we have reported the salient features, which carry the most experimentally accessible signatures. We hope that this will stimulate experimental activity in the field.

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